# Twenty Thousand Leagues under the Words

Combinatorics on words, symbolic dynamics, and S-adic representations

Pierre BÉAUR

# Some context

What is a word?

## A word?

- a (finite) alphabet $\mathcal{A}$ (e.g. $\{a, b\}$)
- a succession of letters $a_0 a_1 \dots a_{n-1}$ (e.g. $abbab$)
- the set of all words: $\mathcal{A}^*$

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What is an infinite word?

## An infinite word?

- a finite alphabet $\mathcal{A}$, still
- the same, but an infinite succession $a_0 a_1 a_2 a_3 a_4 a_5 \ldots$
- set of infinite words: $\mathcal{A}^{\mathbb{N}}$

# Easy infinite words

## Constant word

$aaaaaaaaaaaaaaaaaaaa\ldots$

## Periodic word

$abaaabaaabaaabaaabaa\ldots$

## *Ultimately* periodic word

$baabbba\ abaaabaaabaaabaaabaa\ldots$

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$\Rightarrow$ why are they *easy*?

# Complexity of a word

## A factor

a contiguous subword : $bacb \leq_f abbcbaaabcc\textcolor{red}{bacb}accbaccbbabccca$

## A classical/natural question

How many different factors of length $n$ in $w$ ?

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Remark: $p_n(w) \leq |\mathcal{A}|^n$, exponential bound reached by universal words

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for the periodic word $w = abaaabaaabaaabaaabaaabaaabaa\ldots$

| factor size | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
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## A first result?

If $w$ is (ultimately) periodic, then $(p_n(w))_{n\in\mathbb{N}}$ is bounded.

# My first theorem in combinatorics on words

- a generalization:

**Morse-Hedlund theorem**

$w$ is ultimately periodic $\iff$ there is $n \in \mathbb{N}$ s.t. $p_n(w) \leq n$.

# My first theorem in combinatorics on words

- a generalization:

**Morse-Hedlund theorem**

$w$ is ultimately periodic $\iff$ there is $n \in \mathbb{N}$ s.t. $p_n(w) \leq n$.

a proof:

- $w$ ultimately periodic $\implies p_n(w) \leq n$
- reciprocate:
  - $p_n(w)$ **increasing** with $n$
  - if $p_1(w) = 1$: $w = aaaaaaa\ldots$
  - else, there is $1 < k < n$, $p_k(w) = p_{k+1}(w)$

$\Rightarrow$ **bijection** between {factors of length $k$} and {factors of length $k+1$}

  - let
$$\begin{aligned} \phi\colon \{\text{factors of length } k\} &\to \{\text{factors of length } k+1\} \\ a_0a_1\ldots a_{k-1}a_k &\mapsto a_0a_1\ldots a_{k-1} \end{aligned}$$
  - **conclusion** on the blackboard!

# My favourite words ♡

## Morse-Hedlund theorem

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## Sturmian words

$w$ is **Sturmian** iff $p_n(w) = n + 1$ for all $n$.

## Properties of Sturmian words

- they exist!
- the **simplest aperiodic words** (with many regular properties)
- many combinatorial interpretations!

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- the grid
- $a$ = crossing a vertical line,
  $b$ = horizontal
- (if both: you choose)

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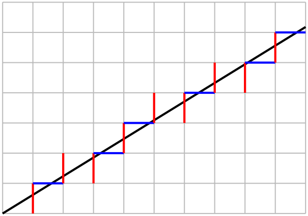

- an irrational slope
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# Now, let's dive

- with finite words: morphisms (= **substitutions**)

## An example of substitution

$\mu : a \mapsto bc, b \mapsto aaa, c \mapsto abc$

$\mu(abca) = \mu(a)\mu(b)\mu(c)\mu(a) = bcaaaabcbc$

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$$\sigma^0(a) = a$$

take $\sigma : a \mapsto ab, b \mapsto a,$

iterate on $a$

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\end{aligned}$$

it builds a longer and longer word

$\Rightarrow$ an **infinite word**?

# A necessary technical point

. . . but it builds a long *finite* word

need a **topology/convergence**

## An idea of the topology

$(w_n)_{n\in\mathbb{N}}$ **converges** towards an infinite word $w$ $\left(w_n \xrightarrow[n\to+\infty]{} w\right)$

if for every **position** $p$ in $w$, there is a moment where every $w_n$ are long enough and all agree on the position $p$.

# An example

$abbb$

$aaab$

$aabaa$

$aaaaaabab$

$aaaaaab$

$aaaababba$

$aaaaaa$

$aaaaaaaab$

$\ldots$

# An example

| | |
|---|---|
| $abbb$ | $\sigma^0(a) = a$ |
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| $aabaa$ | $\sigma^2(a) = aba$ |
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| $aaaaaa$ | $\sigma^6(a) = abaababaabaababaababa$ |
| $aaaaaaaab$ | $\sigma^7(a) = abaababaabaababaababaabaababaabaab$ |
| $\dots$ | $\dots$ |

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| $aaaaaaab$ | $\sigma^7(a) = abaababaabaababaababaabaababaabaab$ |
| ... | ... |

$\Rightarrow$ $(\sigma^n(a))_{n\in\mathbb{N}}$: (here) a prefix sequence of words

- the limit: a **substitutive word**

# When does it make sense?

**Question**

For which substitution $\sigma$ is there a letter $a$ s.t. $(\sigma^n(a))_{n\in\mathbb{N}}) \xrightarrow[n\to\infty]{} w \in \mathcal{A}^\omega$?

**Answer (folk.)**

Iff there is $b$ such that $\sigma(b)$ begins with $b$ (the limit exists), and $b$ can be reached with an expansion (the limit is an infinite word).

- an effective way to compute the prefixes
- a whole family of new words!

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## Some substitutive words

Fibonacci word: □■□□■□■□□■□□■□■□□■□■□□■□□■□■□□■□□■ . . .

- in fact, a Sturmian word ♡

Thue-Morse word: □■■□■□□■■□□■□■■□■□□■□■■□□■■□■□□■ . . .

- not Sturmian: $p_n(w) = 4n$

# Substitutive words

## Combinatorial properties of substitutive words

- fix-points of their substitution
- $p_n(w) = \mathcal{O}(n^2)$
- has a recurrent suffix

## What this means

recurrent: every factor appears infinitely often

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  - $w$ minimal and linearly recurrent

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primitive: a "non-degenerate" substitution

uniformly/linearly recurrent: bounds on the gaps between occurences

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## Historically a counterintuitive word

Thue-Morse word: first example of an aperiodic linearly recurrent word

# Decidable?

## Many questions

Given a substitution (or two):

- is the limit word ultimately periodic?
- is the limit word uniformly recurrent?
- are the two limit words equal?

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### Durand, ~1995-2002

It is all decidable.

### For connoisseurs

- also for morphic words
- as well as isomorphism between minimal subst. systems
- and one system being the factor of another

# We were all rooting for you!

- many results for $\lim_{n\to\infty} (\sigma^n(a))_{n\in\mathbb{N}}$!
- but boring: always iterating the same substitution. . .

**Even worse**

Most Sturmian words are **not** substitutive.

$\Rightarrow$ **let's take a generalization!**

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## $S$-adic representation

Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of substitutions, and $(a_n)_{n\in\mathbb{N}}$ a sequence of letters.
$w$ is generated by the $S$-adic representation $(\sigma_n, a_n)_{n\in\mathbb{N}}$ if

$$w = \lim_{n\to\infty} \sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_n(a_n)$$

. . . what does it mean?

# Explaining the generalization

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  - $L_a : a \mapsto a, b \mapsto ab$
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$L_a(b) = ab$

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$L_aL_bL_aL_aL_bL_a(b) = abaabaababaabaabaab$
$\Rightarrow$ composition is backwards, at the end

# First results

- why are they cool?

**Berstel-Séébold 2002**

There are 4 subs. s.t. $w$ is Sturmian $\iff$ $w$ has an $S$-adic rep. using them.

**Generalized: Arnoux-Rauzy 2007, Gheeraert-Leroy-Lejeune 2021**

Arnoux-Rauzy words and minimal dendric ternary words (= other interesting families of words) have an $S$-adic characterization.

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**Ferenczi 1996**

If a word $w$ is aperiodic, is uniformly recurrent and has a linear complexity, then $w$ admits an $S$-adic representation.

**The $S$-adic conjecture**

Can we find a condition $C$ s.t.
$w$ has an $S$-adic rep. following condition $C$ iff $w$ has linear complexity ?

# A bit too strong?

**Cassaigne 2009**

If $w$ is defined on the alphabet $\mathcal{A}$, then $w$ has an $S$-adic representation using the alphabet $\mathcal{A} \cup \{\mathbf{l}\}$.

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**A limitation (Cassaigne 2009, too)**

If we **forbid cheat letters**, universal words don't have any $S$-adic representations.

# Hi Mom, I'm on TV!

## An $S$-adic system

Wit $\mathfrak{S}$ a set of substitutions, $S_{\mathfrak{S}}$ is the set of words with an $S$-adic rep. using $\mathfrak{S}$.

## Questions

Given a set of substitutions $\mathfrak{S}$:

- is $S_{\mathfrak{S}}$ empty?

Given a second set of substitutions $\mathfrak{T}$:

- is $S_{\mathfrak{S}} \cap S_{\mathfrak{T}}$ empty?

Given a finite $\omega$-automata $\mathfrak{A}$:

- is $S_{\mathfrak{S}} \cap \mathcal{L}(\mathfrak{A})$ empty?

# Hi Mom, I'm on TV!

## An $S$-adic system

Wit $\mathfrak{S}$ a set of substitutions, $S_{\mathfrak{S}}$ is the set of words with an $S$-adic rep. using $\mathfrak{S}$.

## Questions

Given a set of substitutions $\mathfrak{S}$:

- is $S_{\mathfrak{S}}$ empty?

Given a second set of substitutions $\mathfrak{T}$:

- is $S_{\mathfrak{S}} \cap S_{\mathfrak{T}}$ empty?

Given a finite $\omega$-automata $\mathfrak{A}$:

- is $S_{\mathfrak{S}} \cap \mathcal{L}(\mathfrak{A})$ empty?

## B.-Hellouin, draft 2022?

It is decidable!

## Application

I can decide if there is a Sturmian word accepted by an $\omega$-automaton!

- a very old field of research, many links with computation, numeration systems, text algorithms, . . .
- the very (easy) basis of symbolic dynamics
- in general: all's well that ends well
- in search of a difficulty: would mean links with calculability theory

Thank you!

Questions?