Strong local rules without labels for Penrose rhombus tilings

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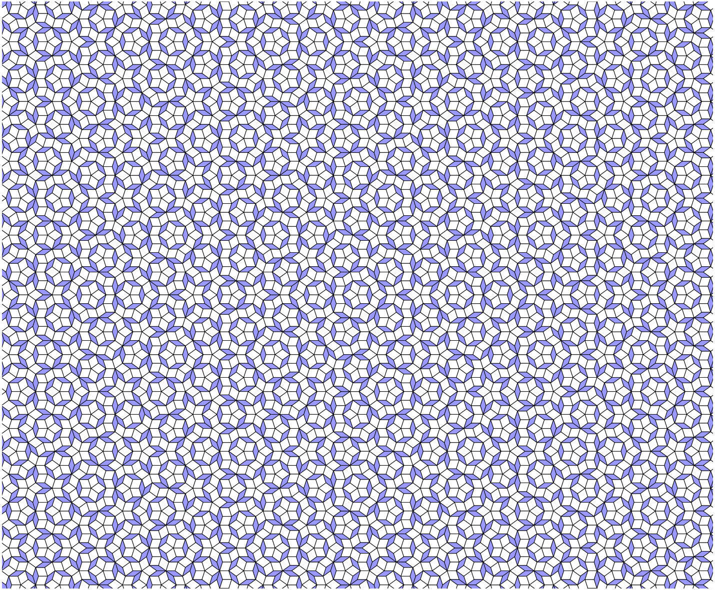
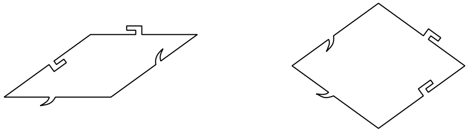
with Thomas Fernique

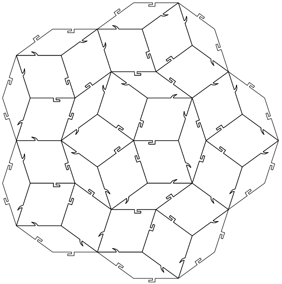
2022

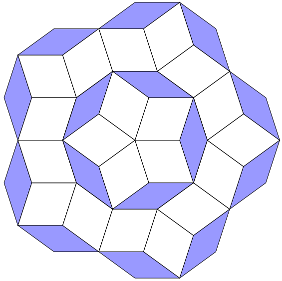
Penrose rhombus tiling



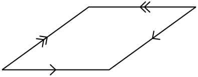
Original definition [Penrose, 1974]

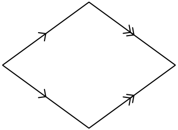


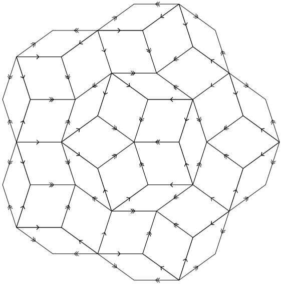


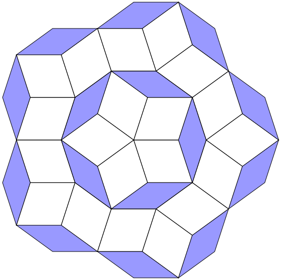


Improved definition [De Bruijn, 1981]









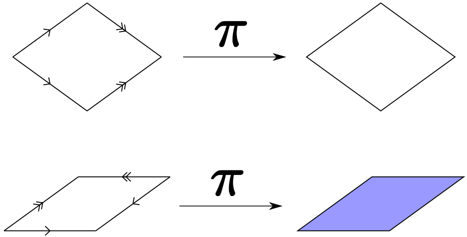
Label simplification

A bit of formalism:

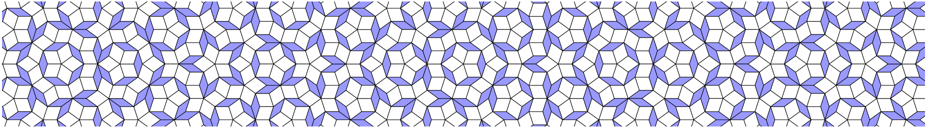
- geometrical tile : t a compact of $\mathbb{R}^2$, here a rhombus,
- labelled tile : (t, l) with t a geometrical tile, and l a function from t to a finite alphabet $\mathcal{A}$, label condition: for any two tiles (t, l) and (t', l'), we have $\forall x \in t \cap t',\ l(x) = l'(x)$.

The label simplification π is defined as $\pi(t, l) = t$.

$X_a :=$ tilings with the arrow-labelled tiles,
$X_p := \pi(X_a)$ its projection.



First properties [Grünbaum and Shephard, 1987]



The subshift of geometrical Penrose rhombus tilings X_p is defined as:

$$X_p := \pi(X_a)$$

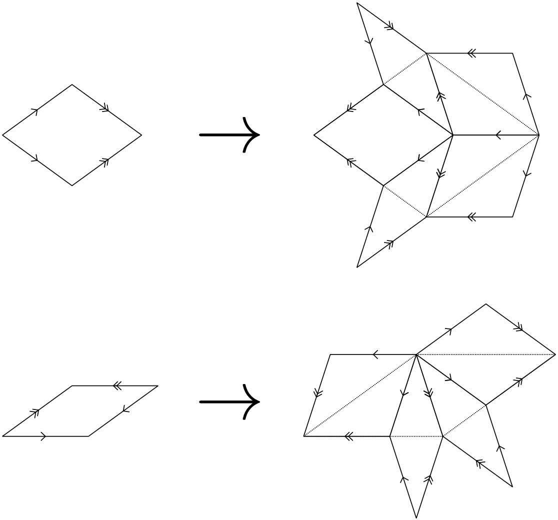
First properties:

1. X_p is non empty.
2. X_p is aperiodic *i.e.* does not contain any periodic tilings.

Remark

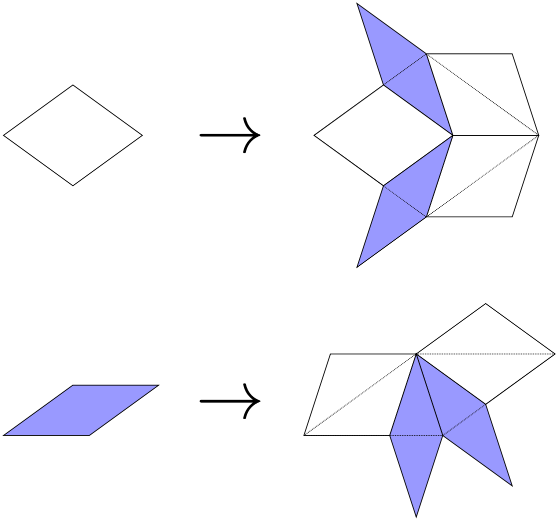
We call this a sofic aperiodic subshift *i.e.* the projection of a SFT.

Substitution



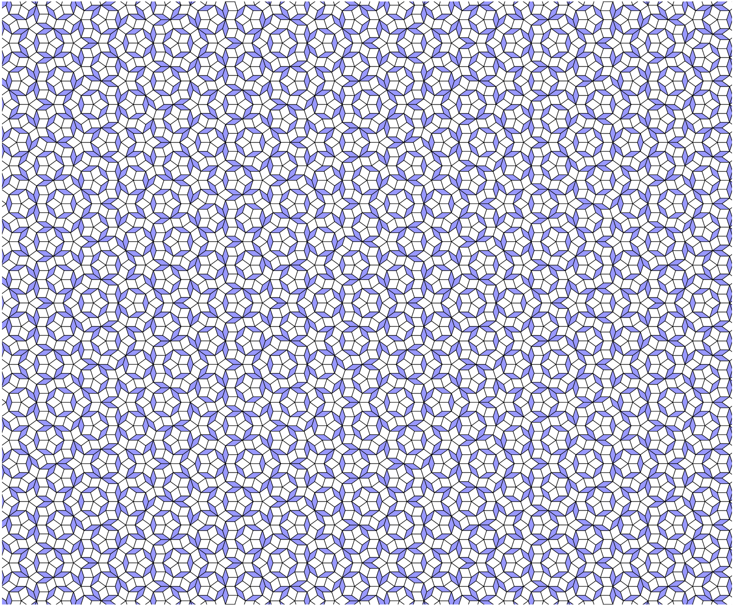
$$X_p = \pi(X_\sigma)$$

Substitution



$$X_p = X_\sigma$$

Substitution, metatiles and decomposition



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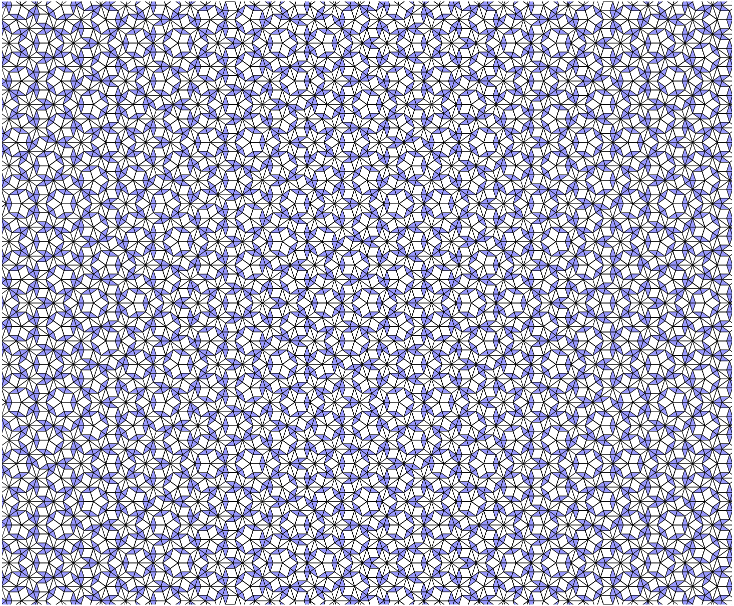

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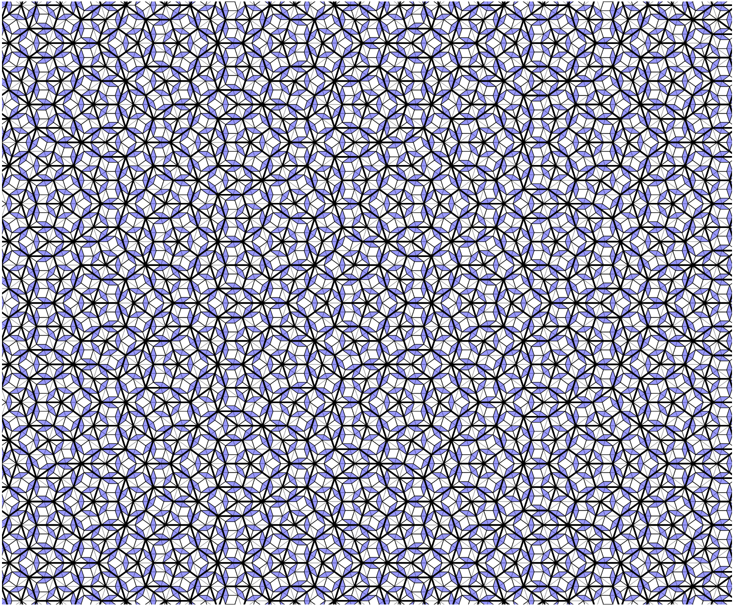

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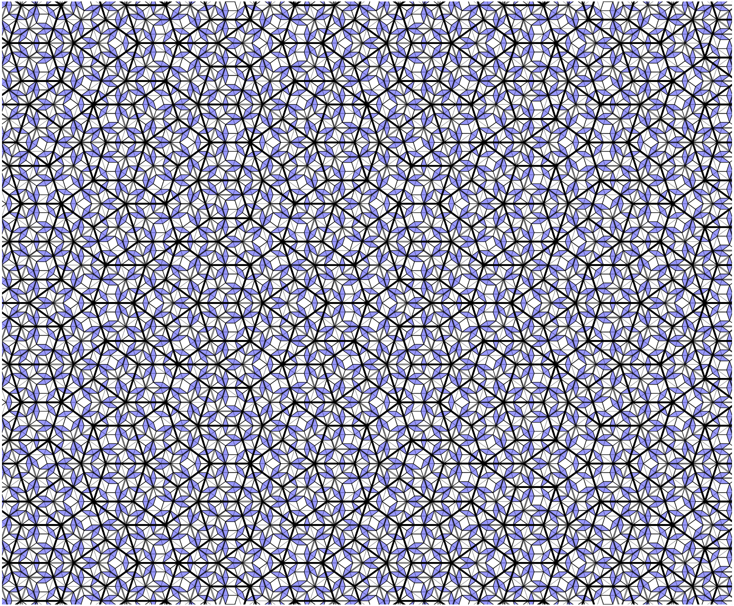
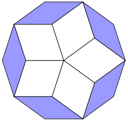


Canonical Penrose tiling: limit tiling from a seed



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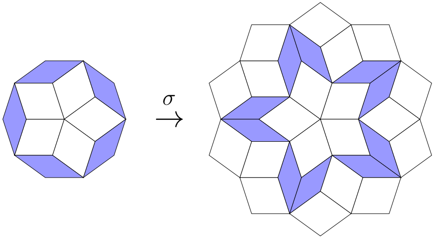

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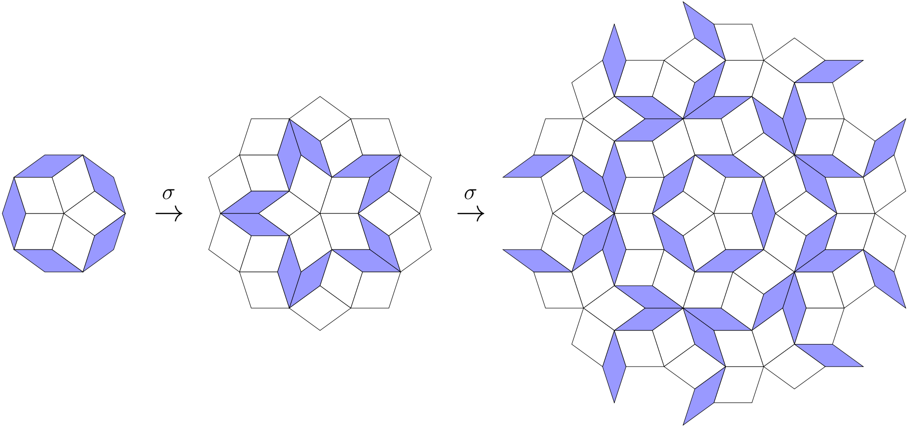
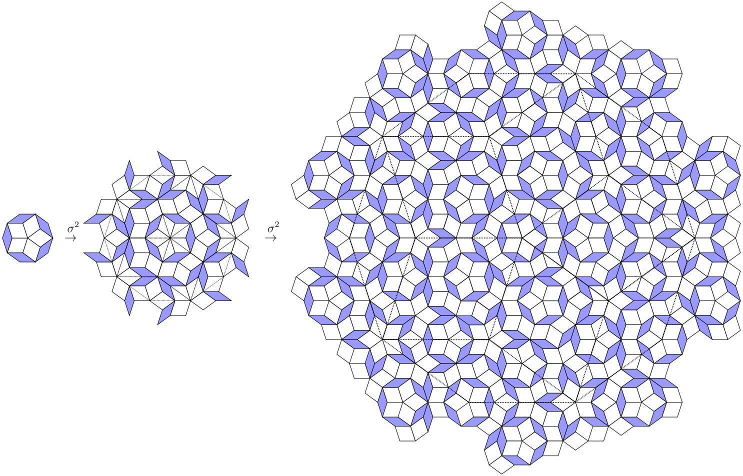


By compacity there exists a Penrose tiling of the full plane.

Canonical Penrose tiling: limit tiling from a seed



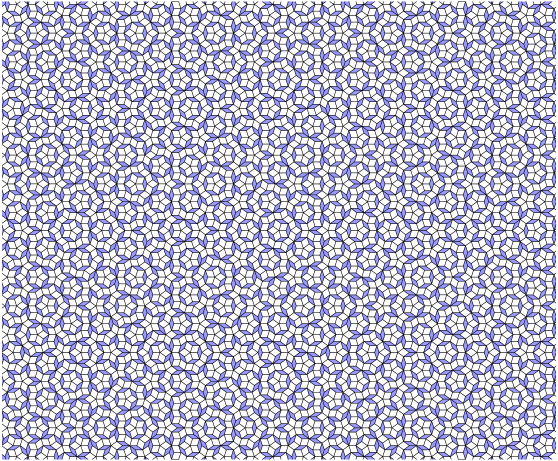
$$\mathcal{T} := \lim_{n \to \infty} \sigma^{4n}(S)$$

Canonical Penrose tiling and rotational symmetry

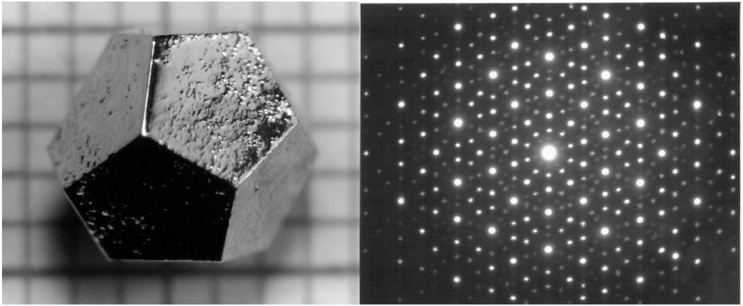
Crystallographic restriction: periodic tilings can only have 2, 3, 4 or 6-fold rotational symmetry.

$\Rightarrow$ a tiling with 10-fold rotational symmetry is non-periodic.

The Penrose tilings have local 10-fold rotational symmetry, so they are non-periodic.

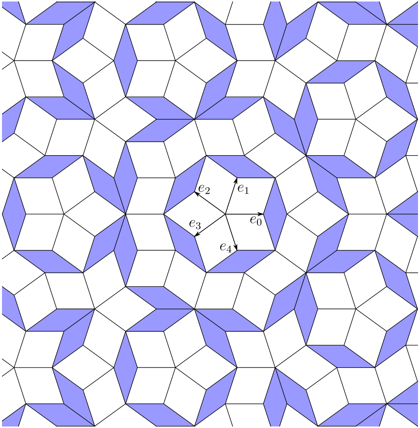


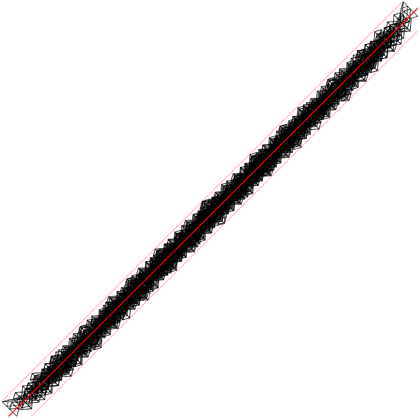
Quasicrystal tiling [Senechal, 1996]



Quasicrystal : sharp diffraction pattern, but non-periodic.

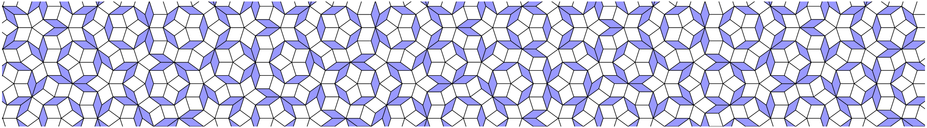
Quasicrystal tilings





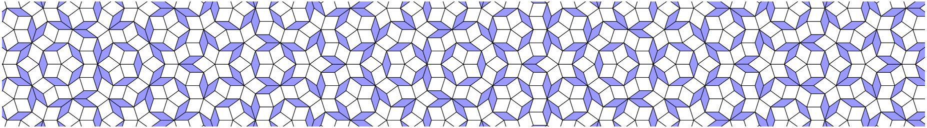
Two models for quasicrystal tilings:
Discrete plane : lifted in $\mathbb{R}^n$ it approximates a plane.
Cut-and-project : lifted in $\mathbb{R}^n$ it strongly approximates a plane :
its vertex set is exactly $(\mathcal{E} + \Omega)\,\mathbb{Z}^n$ where $\mathcal{E}$ is the plane and Ω a compact.

Penrose tiling overview [Baake and Grimm, 2013]



The Penrose subshift X_p is :

- an aperiodic subshift with 10-fold local symmetry
- a sofic subshift $X_p = \pi(X_a)$
- a substitution subshift $X_p = X_\sigma$
- a subshift of quasicrystal tilings



Today's result

Theorem (folk., Fernique-L. 2022)

There exists a finite set of local rules without labels for Penrose rhombus tilings.

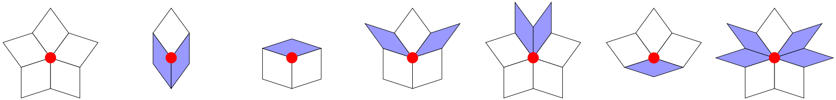
$\Rightarrow X_p$ is a Subshift of Finite Type.

Remark

The local rules will be given by a set of allowed patterns called vertex atlas instead of a set of forbidden patterns.
Note that for tilings with finite local complexity a (finite) vertex-atlas is equivalent to a finite set of forbidden patterns.

Local rules: vertex-atlas

- $\mathcal{N}_{\mathcal{T}}^{k}(x)$: the k-neighbourhood of a vertex x in a tiling is the patch of tiles that are at edge-distance at most k from x.
  For example the 0-neighbourhood of a vertex x is the patch of tiles that directly touch x.



- $\mathbf{A}_k$: a k-vertex atlas is a set of k-neighbourhoods.

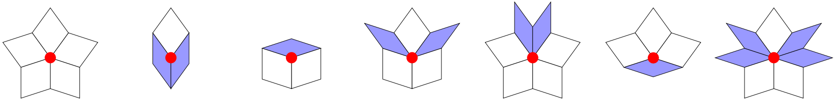
- $\mathcal{N}_{\mathcal{T}}^{k}$: the set of all k-neighbourhoods of the tiling $\mathcal{T}$ (or subshift X).

- $X_{\mathbf{A}}$: subshift of all the tilings $\mathcal{T}$ such that $\mathcal{N}_{\mathcal{T}}^{k} \subseteq \mathbf{A}$.

0-atlas

Definition

Let us define $\mathbf{A}_0$ as pictured below up to isometry



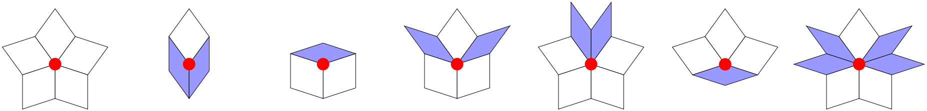
Proposition (folk.)

With X_p the Penrose subshift we have
$$\mathcal{N}^0_{X_p} = \mathbf{A}_0.$$

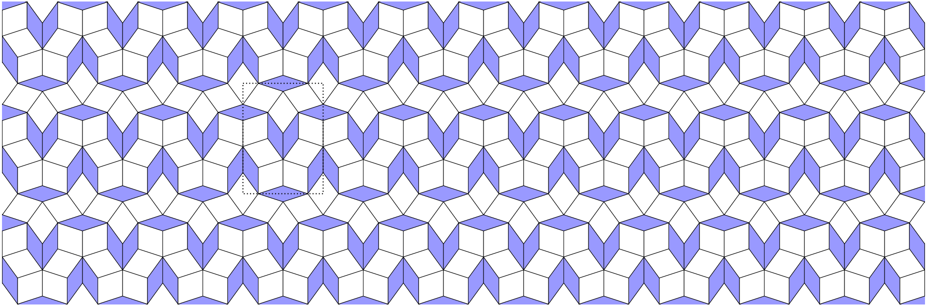
Theorem

With X_p the Penrose subshift we have
$$X_p \subsetneq X_{\mathbf{A}_0},$$
i.e. there are tilings that have the same 0-neighbourhoods as Penrose but are not Penrose tilings.

Proof

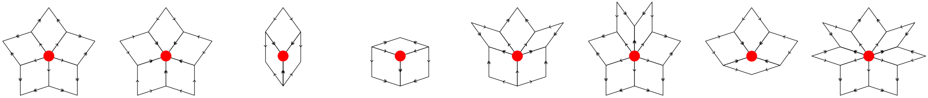


There exists a periodic tiling in $X_{\mathbf{A}_0}$, however X_p is an aperiodic subshift so $X_p \subsetneq X_{\mathbf{A}_0}$.

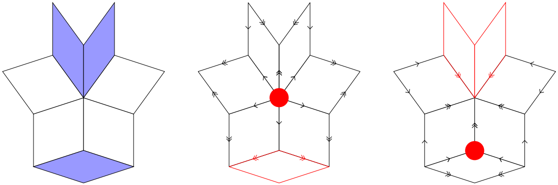


The key point of why it does not work (and why some people think it does).

The patterns in $\mathbf{A}_0$ admit a unique valid labelling, except the 5-star that admits two.



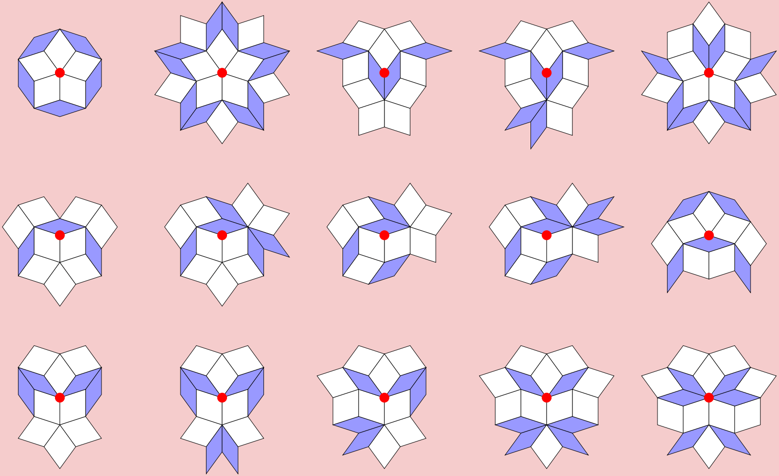
However that does not mean that a pattern that is valid for $\mathbf{A}_0$ admits a valid labelling, the labelling "does not propagate" :



Penrose's 1-neighbourhoods

Proposition (1-neighbourhoods)

With X_p the Penrose subshift we have $\mathcal{N}^1_{X_p} = \mathbf{A}_1$ (pictured here up to isometry)

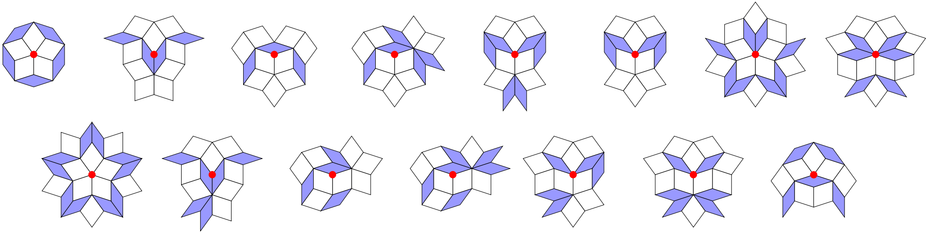


The 1-atlas defines Penrose

Theorem

With X_p the Penrose subshift, and with the 1-atlas $\mathbf{A}_1$ defined in the previous proposition we have

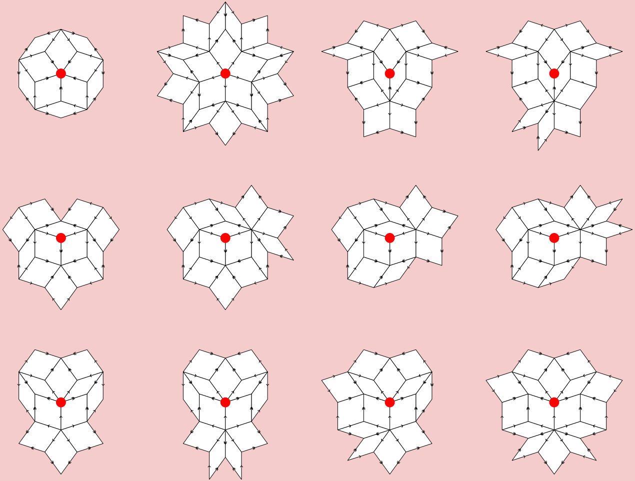
$$X_{\mathbf{A}_1} = X_p.$$



Recall that initially the Penrose subshift is defined as a sofic subshift with $X_p := \pi(X_a)$, now with this characterisation $X_p = X_{\mathbf{A}_1}$ we obtain that X_p is a Subshift of Finite Type (SFT).

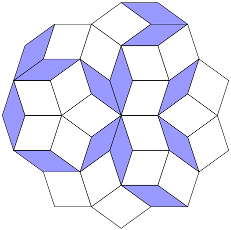
Proposition

The patterns of $\mathbf{A}_1$ have a unique labelling:



Lemma

Let P be an edge-connected patch of geometrical Penrose tiles. Let t be a tile in P. Let $l(t)$ be a Penrose labelling of the tile t.
There exists at most one valid Penrose labelling $\mathcal{L}$ of P such that $\mathcal{L}(t) = l(t)$.



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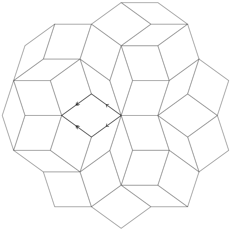

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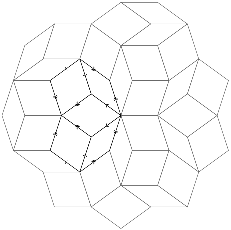

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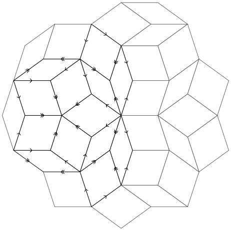

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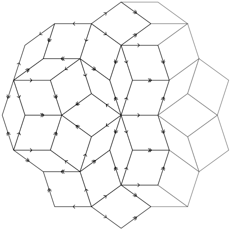

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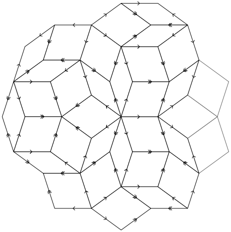

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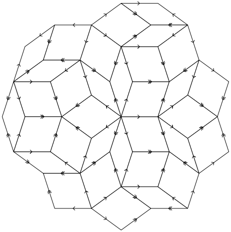

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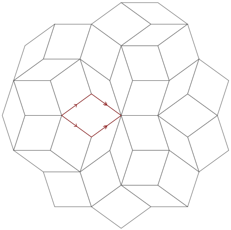

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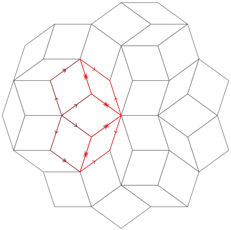

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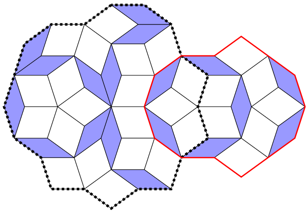
Let P_1 and P_2 be two edge-connected patches of geometrical Penrose tiles such that :

- *$P_1 \cup P_2$ is a patch (i.e. simply connected set of non-overlapping tiles)*
- *$P_1 \cap P_2$ is non-empty and edge-connected*
- *$P_1 \setminus P_2$ is not edge connected to $P_2 \setminus P_1$*
- *P_1 has a valid Penrose labelling $\mathcal{L}_1$*
- *P_2 has a valid Penrose labelling $\mathcal{L}_2$*

If there exists a tile t in $P_1 \cap P_2$ such that $\mathcal{L}_1(t) = \mathcal{L}_2(t)$ then there exists a valid Penrose labelling $\mathcal{L}$ of $P_1 \cup P_2$.



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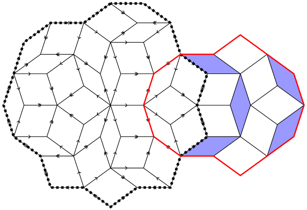

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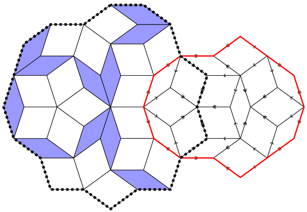

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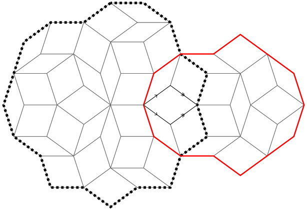

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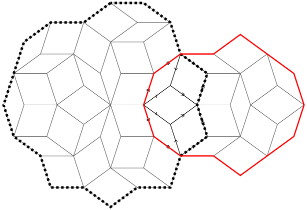
- $P_1 \cap P_2$ *is non-empty and edge-connected*
- $P_1 \setminus P_2$ *is not edge connected to* $P_2 \setminus P_1$
- P_1 *has a valid Penrose labelling* $\mathcal{L}_1$
- P_2 *has a valid Penrose labelling* $\mathcal{L}_2$

If there exists a tile t *in* $P_1 \cap P_2$ *such that* $\mathcal{L}_1(t) = \mathcal{L}_2(t)$ *then there exists a valid Penrose labelling* $\mathcal{L}$ *of* $P_1 \cup P_2$.



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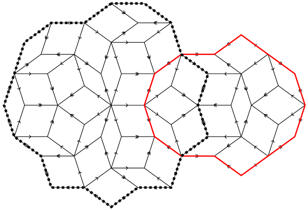

We call 1-interior vertices of P the vertices of which the 1-neighbourhood is complete in P. We say that a finite patch P is *exact* for $\mathbf{A}_1$ if, with $V_1(P)$ the set of 1-interior vertices of P, $V_1(P)$ is connected, we have $\mathcal{N}_P^x(1) \in \mathbf{A}_1$ for all $x \in V_1(P)$ and P is exactly the union of the 1-neighbourhoods.

Proposition

Let P be an exact patch for the vertex-atlas $\mathbf{A}_1$.
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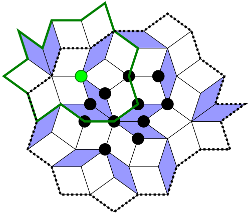
Proof.

By induction on the number of 1-interior vertices.
1 : an exact patch with 1 1-interior vertex is exactly a 1-neighbourhood, *i.e.* a patch in $\mathbf{A}_1$ up to isometry, these patch have a valid Penrose labelling.
$\mathbf{n} \to \mathbf{n+1}$: we can decompose P_{n+1} as $P_n \cup P$ with P_n an exact patch (for $\mathbf{A}_1$) with n 1-interior vertices and P an exact patch with 1 1-interior vertex. We apply the previous lemma.



Note that the fact that both P_{n+1} and P_n are exact patches is a strong condition. This means that P is the 1-neighbourhood of a *suitably chosen* vertex $v \in V_1(P_{n+1})$. □

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Let $\mathcal{T}$ be a valid tiling for $\mathbf{A}_1$.
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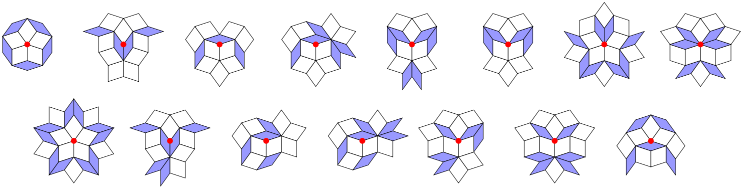
Proof.

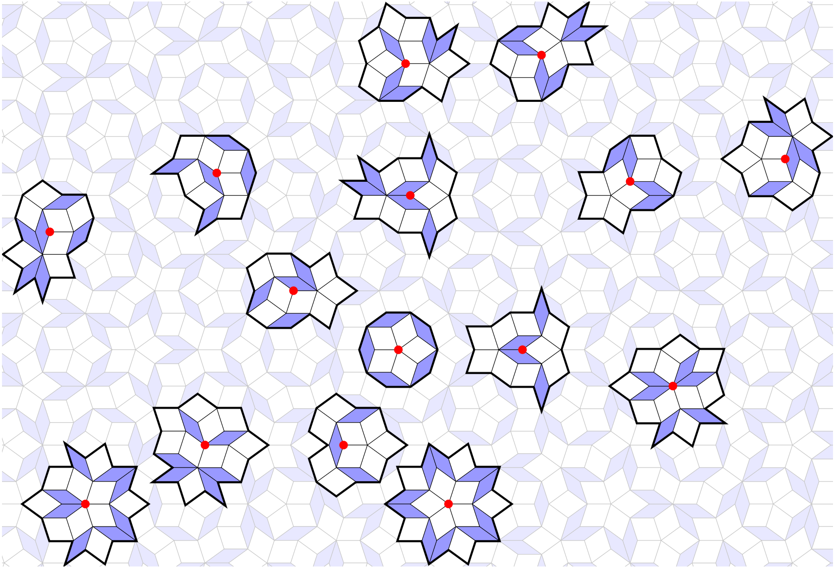
We can build a sequence of increasing (non-empty) exact patches $(P_n)_{n\in\mathbb{N}}$ that tends to the whole tiling.
For all n we have $P_n \subseteq P_{n+1}$, and P_n has a unique valid Penrose labelling. So the labelling of P_{n+1} extends the labelling of P_n. This gives us a unique labelling for the whole tiling. □

The hidden difficulty

For now we have actually proved that $X_{\mathbf{A}_1} \subseteq X_p$ *i.e.* we have proved that any tiling that is legal for $\mathbf{A}_1$ is a Penrose tiling.
However this does not prove that $X_{\mathbf{A}_1}$ is non-empty, and it does not prove either that $X_p \subseteq X_{\mathbf{A}_1}$ *i.e.* that Penrose tilings are legal for $\mathbf{A}_1$.
To prove that we need to prove the proposition on Penrose's 1-neighbourhoods *i.e.* we need to prove that $\mathcal{N}^1_{X_p} = \mathbf{A}_1$.



First inclusion $\mathbf{A}_1 \subseteq \mathcal{N}^1_{X_p}$



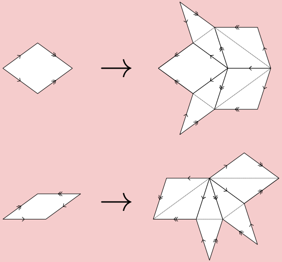
Second inclusion $\mathcal{N}^1_{X_p} \subseteq \mathbf{A}_1$

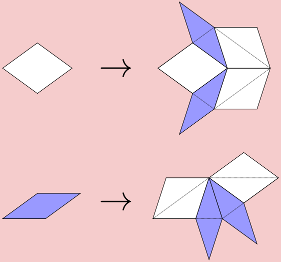
Hard to prove.

Goal: use know properties of X_p to find a finite fragment of tiling which contains all the 1-neighbourhoods.

Theorem (substitution)

Penrose's rhombus tilings are substitution tilings.





Linear recurrence

Uniformly recurrent: for any pattern P, there exists a diameter D such that P appears in any disk of diameter D of the tiling.

Linearly recurrent: there exists a constant C such that for any pattern P, P appears in any disk of diameter $C \cdot \mathrm{diam}(P)$ of the tiling.

Similitude substitution: a vertex-hierarchic substitution where the expansion is a direct similitude of the plane.

Theorem ([Solomyak, 1998])

Primitive similitude substitution tilings are linearly recurrent.
Moreover the linear recurrence factor C is bounded by

$$C \leqslant \frac{C_0 \cdot \lambda}{C_1}$$

where λ is the scaling factor of the substitution, C_0 is the appearance radius of the 0-patterns, and C_1 is the minimum inner diameter of the tiles.

Note that the existence of C_0 is a consequence of the primitivity of the substitution.
Note also that if the substitution is not a similitude substitution we only have uniform recurrence.

An upper bound on Penrose's linear recurrence factor

$$C \leqslant \frac{C_0 \cdot \lambda}{C_1}$$

1. λ is the scaling factor of the Penrose substitution : $\lambda = \frac{1+\sqrt{5}}{2} \approx 1.618$

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   - $r_v = \lambda^3 \cdot r_1 = \lambda^3 \cdot \frac{1}{2\sin(\frac{3\pi}{10})} = \lambda^2 \approx 2.618$

     where r_1 is the maximum distance from a point of $\mathbb{R}^2$ to a vertex in a Penrose tiling.
   - $r_c = \sqrt{a^2 + b^2 - 2 \cdot a \cdot b \cdot c} \approx 6.613$
     with $a = 2 + 4\cos\frac{\pi}{5} + 2\cos\frac{2\pi}{5}$, $b = 2\cos\frac{3\pi}{10}$ and $c = \cos\frac{7\pi}{10}$.
     Here r_c is the radius of appearance of the 0-patterns up to isometry in the 3rd image of the 0-patterns.

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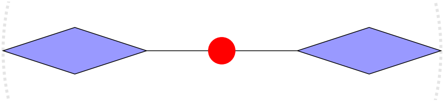
$\Rightarrow$ Penrose's linear "up-to-isometry" recurrence factor $C \leqslant \frac{C_0 \cdot \lambda}{C_1} \leqslant 25.414$.

An upper bound on the appearance radius of 1-neighbourhoods

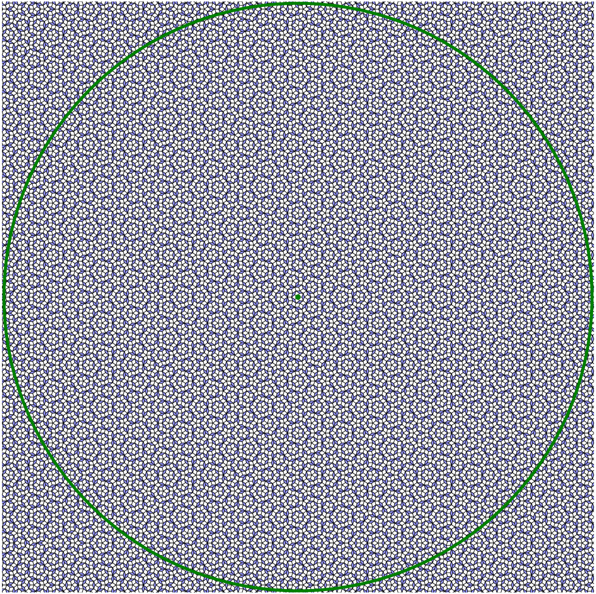
We have $C \leqslant 25.414$.
The maximum possible diameter of a 1-pattern with geometrical Penrose rhombus tiles is $2(1 + 2\cos\frac{\pi}{10})$.



$\Rightarrow$ all 1-patterns of Penrose tilings appear up to isometry in any patch of diameter $\mathcal{D}_{\mathbf{A}_1}$ with

$$\mathcal{D}_{\mathbf{A}_1} := 2(1 + 2\cos\tfrac{\pi}{10}) \cdot C \leqslant 147.51.$$

The fragment that contains all 1-neighbourhoods



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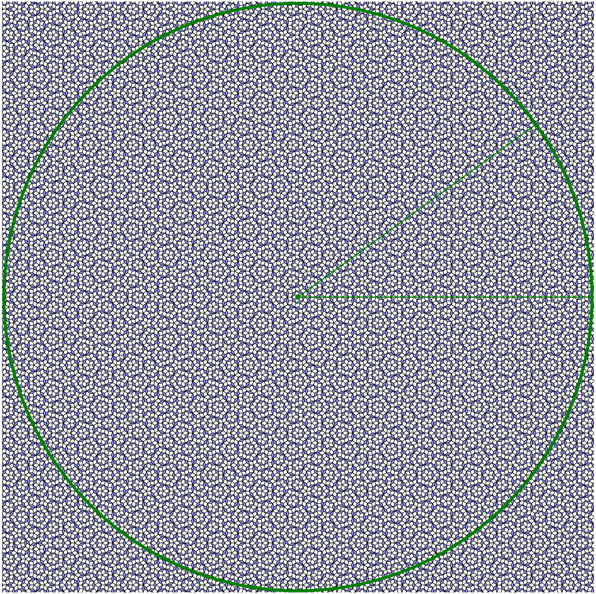

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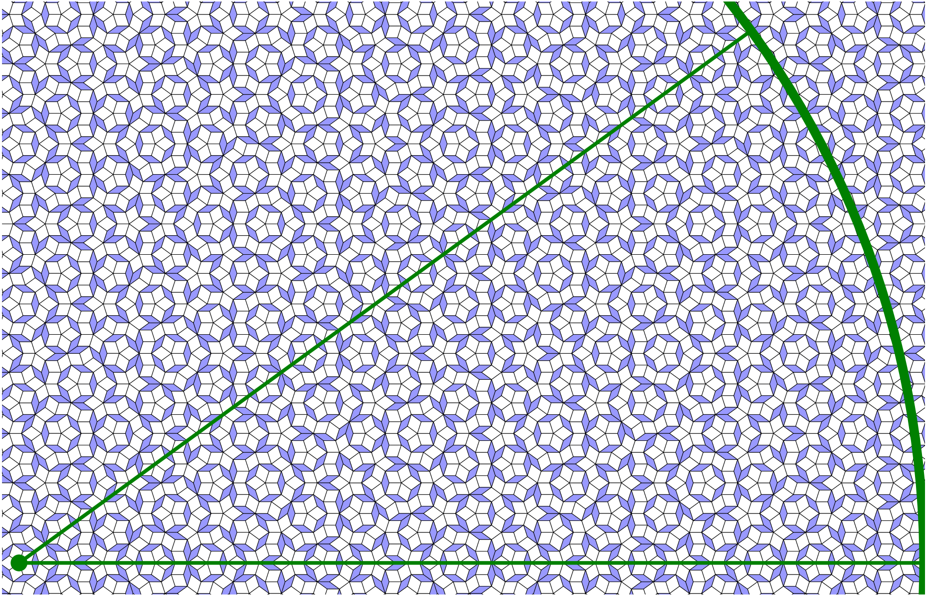
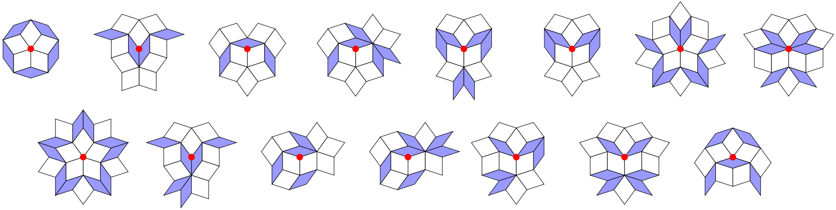


Conclusion

1. the 0-atlas does not characterise the geometrical Penrose rhombus tilings X_p,
2. the 1-atlas characterises the geometrical Penrose rhombus tilings X_p, in particular this means that X_p is a Subshift of Finite Type.
3. the 1-atlas $\mathbf{A}_1 := \mathcal{N}^1_{X_p}$ is :



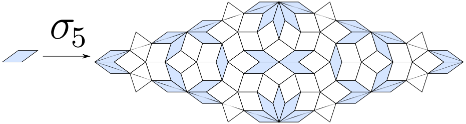
(up to isometry).

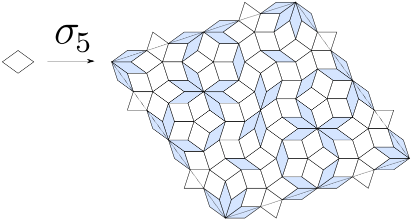
Planar rosa tilings

The (canonical) Penrose rhombus tiling is both a quasicrystal tiling and substitution tiling with global 5-fold rotational symmetry.
Question: do substitution quasicrystal tilings exist for any order of rotationaly symmetry?

Theorem ([Kari and Lutfalla, 2021] [Kari and Lutfalla, 2022])

- the Sub Rosa tilings [Kari and Rissanen, 2016] are not quasicrystal tilings.
- the Planar Rosa tilings are substitution quasicrystal tilings with $2n$-fold rotational symmetry.



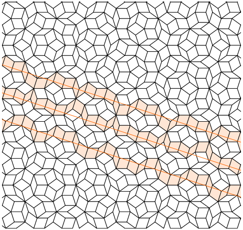


Future work:

- determine if the Planar Rosa tilings are cut-and-project.
- characterise the slopes of substitution quasicrystal tilings.

The multigrid method

The (canonical) Penrose rhombus tiling is $\mathcal{P}_5(\frac{1}{5})$, the 5-fold multigrid dual tiling with offset $\frac{1}{5}$.



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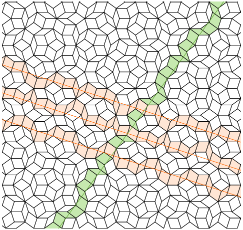

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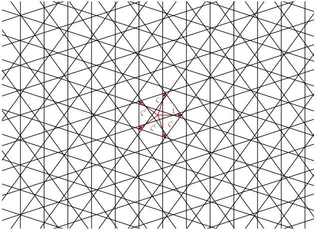

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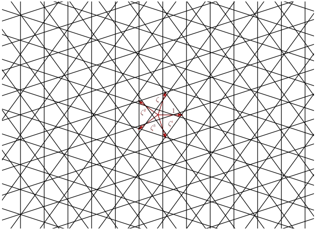

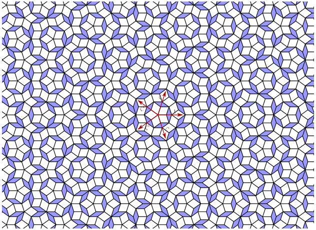

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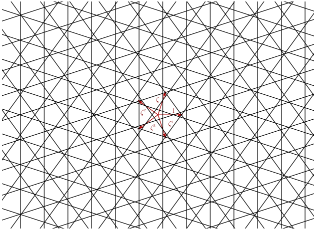

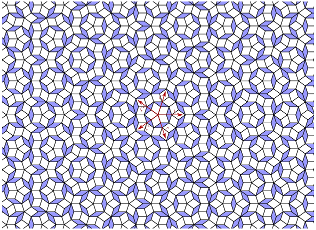

Theorem ([Lutfalla, 2021])

- For any **odd** n, the n-fold multigrid dual tiling $\mathcal{P}_n(\frac{1}{n})$ is a rhombus cut-and-project tiling with n-fold rotational symmetry.
- For any n, the n-fold multigrid dual tiling $\mathcal{P}_n(\frac{1}{2})$ is a rhombus cut-and-project tiling with $2n$-fold rotational symmetry.

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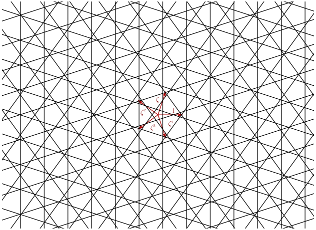

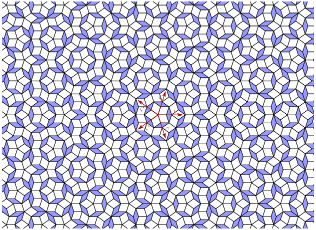

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$X_5 :=$ subshift of 5-fold multigrid dual tiling.
We know that X_5 is not minimal and that $X_p \subsetneq X_5$.
Future work: study the decomposition of X_5 in minimal subshifts.

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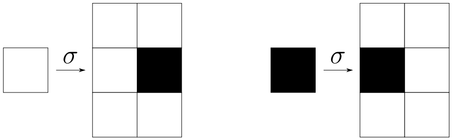
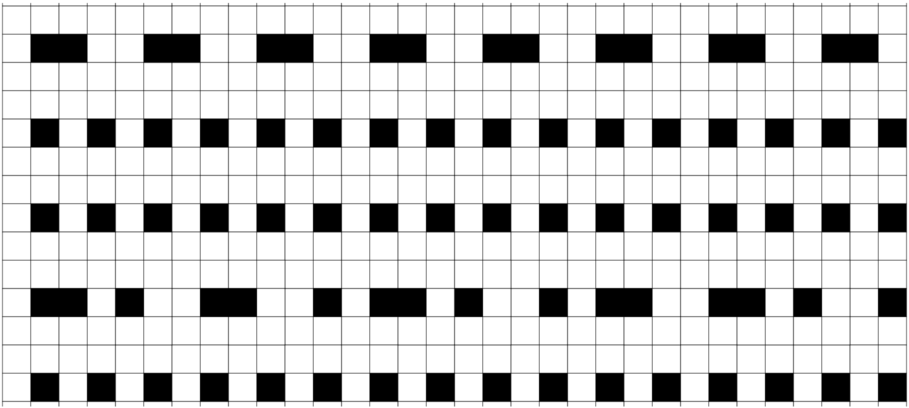
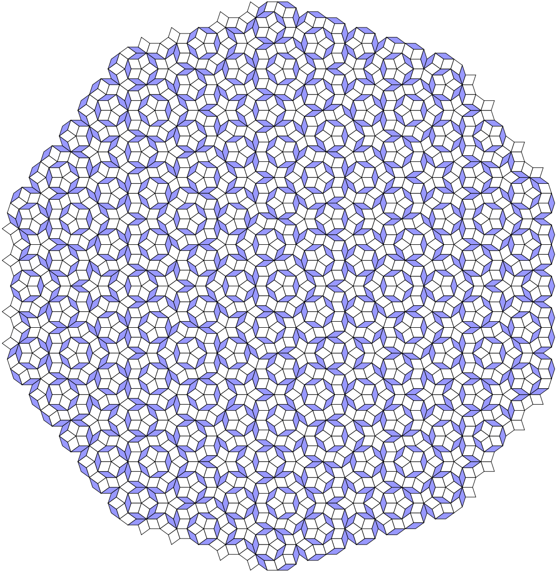
General substitution and linear recurrence a counterexample





Substitution and linear recurrence idea of the proof

Penrose as a cut-and-project



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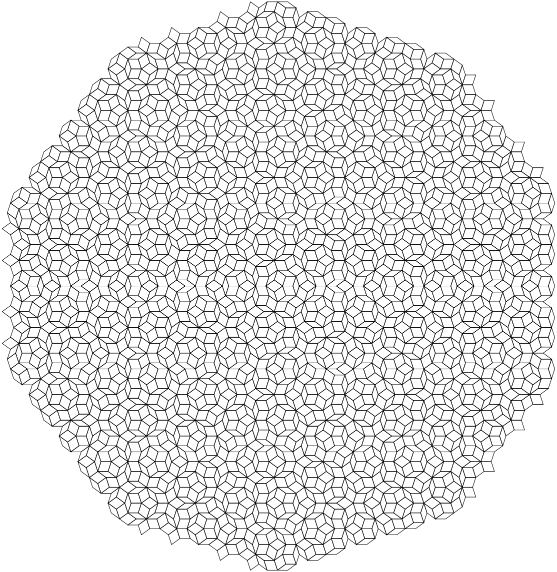

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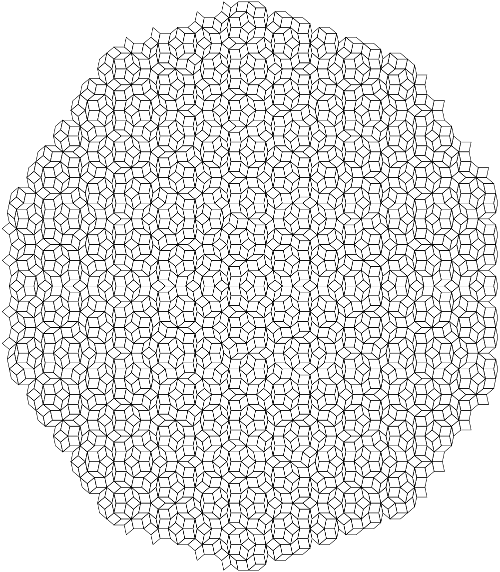

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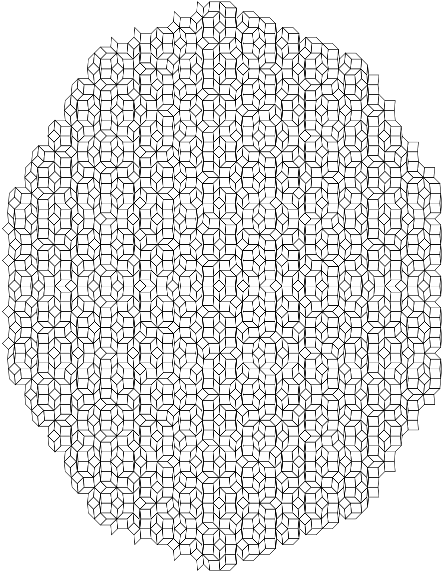

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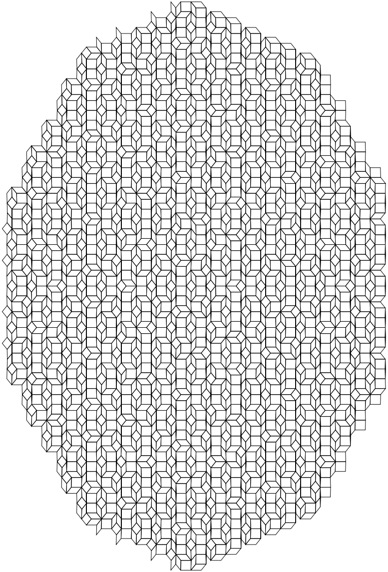

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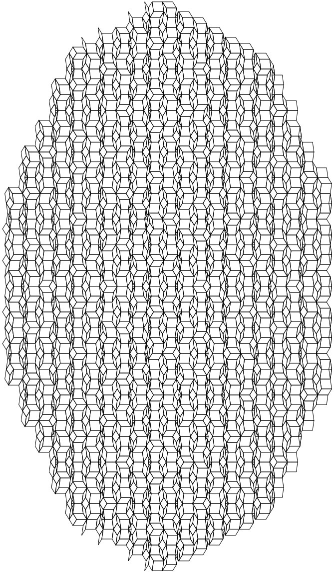
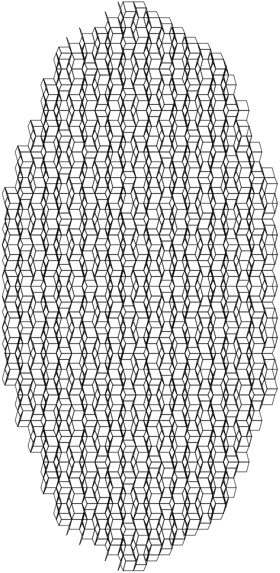


Penrose as a cut-and-project



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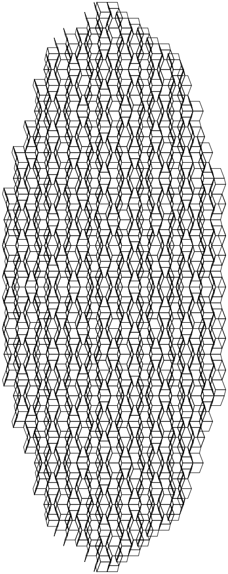

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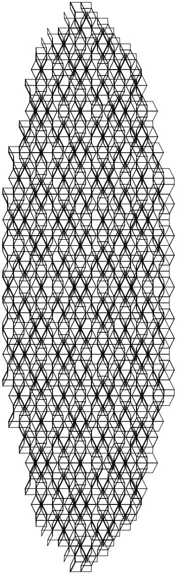

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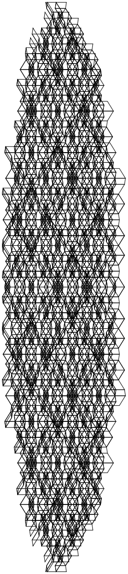

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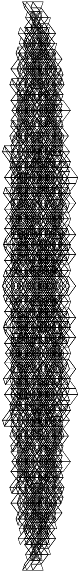
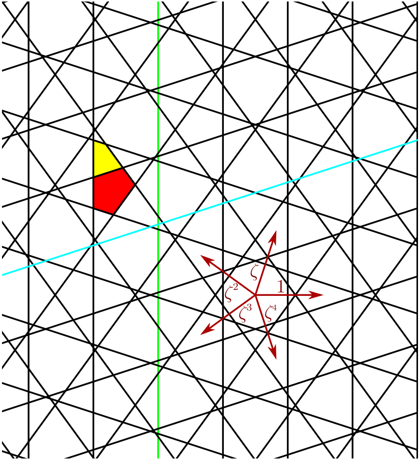
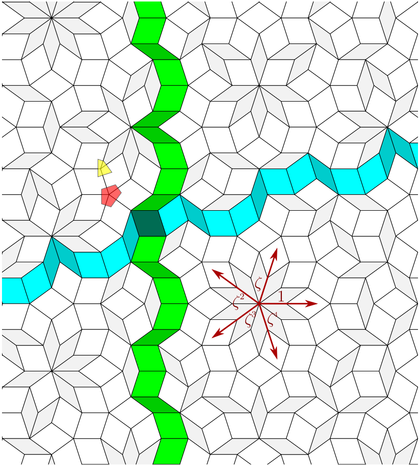


Penrose as a cut-and-project

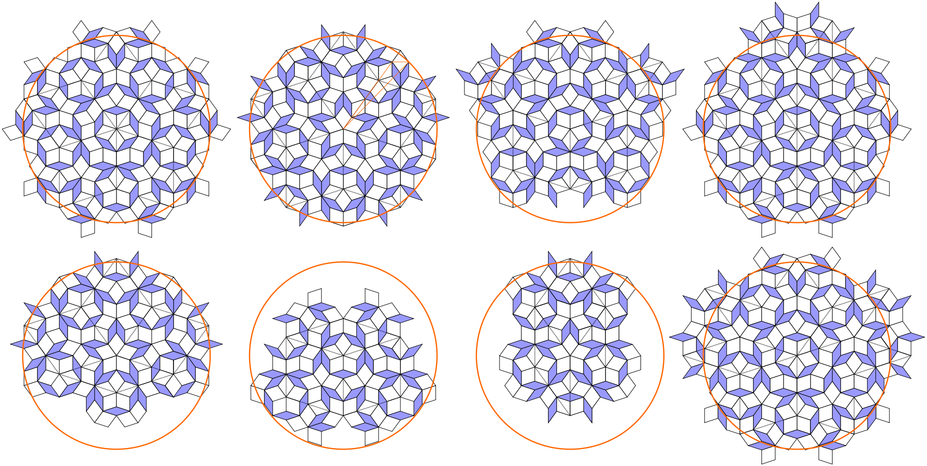
Antipenrose





Computing C_0

$r_c = \sqrt{a^2 + b^2 - 2 \cdot a \cdot b \cdot c} \approx 6.613$
with $a = 2 + 4\cos\frac{\pi}{5} + 2\cos\frac{2\pi}{5}$, $b = 2\cos\frac{3\pi}{10}$ and $c = \cos\frac{7\pi}{10}$.



Forbidden patterns